## Contents



Cracking Bertrand's paradox

- About the theory of Measure and Probability

## References


[1] Jacques Neveu, "Mathematical foundations of the calculus of probability" translated from, Bases Mathématiques du Calcul des Probabilités, 1964, Masson et Cie, Paris © Copyright 1965 by Holden-Day, Inc., 728 Montgomery Street, San Francisco, California.

[2] Laurent Schwartz | Analyse | volume=I, III and IV | publisher=Hermann Editor | Paris, France | October 21, 1997 | isbn=978-2705661618


# I Incipit

Here, we review the so-called Bertrand's paradox. This paradox is illustrated in the first three examples in: Link to wikipedia: https://en.wikipedia.org/wiki/Bertrand_paradox_(probability).

These paradoxes arise from the assumption that probability, in each case, has a uniform density. However, as demonstrated in Example 6, this assumption is not dictated by the way we define a chord. It is merely based on a plausible guess about probability density, yet it fails to meet expectations given the inconsistencies it generates.

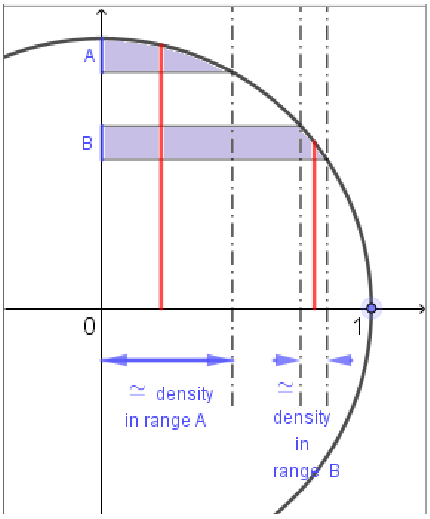
More broadly, the reasoning always invoked —though left unspoken and conveniently brushed aside— is: "Thanks to equiprobability, the probability density must be constant". However, there is no solid evidence to support this claim. In fact, in Example 6 the density cannot be constant.

Looking closely at Example 2 (See figure: 1), for instance, one can observe that chords tend to cluster near the center, exhibiting similar lengths, while their density in relation to their length decreases as they shrink to

 Author: Camille DUVIVIER – camille.duvivier.fr@gmail.com – review: May 10, 2025

zero. The density is significantly higher near the center of the disk than near its boundary. Thus, assuming a uniform density along the radius seems rather far-fetched.

Figure 1: Density along the radius in example 2




Bertrand's precept, which states that probability rarely presents itself as an obvious or intuitive concept, is more relevant than ever. If we assume that the laws of probability are self-evident or straightforward and rely too much on intuition, we risk treading on quicksand, inevitably leading to paradoxes and unexpected contradictions.

## II Abstract

We present calculations for Examples 1., 2., and 3. of Bertrand's paradox
(see Link to wikipedia: https://en.wikipedia.org/wiki/Bertrand_paradox_(probability) for a simple rough description of these examples)
and introduce three additional examples in the same vein, highlighting the flaws in such calculation strategies.

We then demonstrate that Bertrand's conundrum actually has a unique solution.

Finally, we propose a method to properly address the problem and carry out the calculations accordingly.

## III Preliminaries

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ref: [1, 2]
You may skip this section and overlook it with regard to the following examples 1. to 6. from chapters IV to IX.

$\|.\|$ designates the Euclidean norm.
The unit circle $S_1 = \{u \in \mathbb{R}^2 / \|u\| = 1\}$ is homeomorphic to $\mathbb{R}/\mathbb{Z}$. On $\mathbb{R}/\mathbb{Z}$ we use the measure $\delta$ of density

1, which corresponds to the length of an arc divided by $2\pi$ modulo 1, on $\mathbb{R}/\mathbb{Z}$.

More formally:
We have the following commutative diagram.

$$\begin{tikzcd}
\mathbb{R} \arrow[d, "\Pi"'] \arrow[dr, "\phi"] & \\
\mathbb{R}/\mathbb{Z} \arrow[r, "\varphi"] & S_1
\end{tikzcd}$$

Where $\phi(\theta) = \exp(2\pi i\theta)$ $\left(\mathbb{C} \sim \mathbb{R}^2,\ \ i^2 = -1\right)$, $\varphi$ homeomorphism[1], $\Pi$ the canonical projection.

We define an arc as the image of a continuous injective function, $\gamma : [0,\ \ 1] \longrightarrow \mathbb{R}/\mathbb{Z}$. And we denote $\mathcal{T}_{\text{arc}}$ the $\sigma$-algebra generated by the arcs of $\mathbb{R}/\mathbb{Z}$.

$p$ the restriction of $\Pi$ on $[0,\ \ 1[$, $p = \Pi_{|[0,1[}$ is bijective:

- $p$ is surjective: $p([0, 1[) = \varphi^{-1}\text{o}\phi([0, 1[) = \varphi^{-1}(S_1) = \mathbb{R}/\mathbb{Z}$.
- $p$ is injective: $x,\ y \in [0, 1[$ such that $p(x) = p(y)$, $\Pi(x) = \Pi(y)$, $x - y \in \mathbb{Z}$, $x = y$.

$\Pi$ is continuous $\left(F \text{ a closed set of } \mathbb{R}/\mathbb{Z},\ \Pi^{-1}(F) = \phi^{-1}\text{o}\varphi(F),\ \ F \text{ closed in compact, compact, } \varphi(F) \text{ compact hence closed, eventually, } \phi^{-1}\left(\varphi(F)\right)\text{closed}\right)$, and so the restriction of $\Pi$ to $]0,\ \ 1[$ also and $p$ is but maybe in 0.
Then $\delta = \mu\text{o}p^{-1}$ is a measure on $(\mathbb{R}/\mathbb{Z},\ \ \mathcal{T}_{\text{arc}})$, $\mu$ the Borel-Lebesgue measure.

And since $\delta(\mathbb{R}/\mathbb{Z}) = \mu\text{o}p^{-1}(\mathbb{R}/\mathbb{Z}) = \mu([0,\ \ 1[) = 1$, $(\mathbb{R}/\mathbb{Z},\ \ \mathcal{T}_{\text{arc}},\ \ \delta)$ is a probability space. And our best bet is to choose it as the probability space of the experiment consisting of picking randomly a point on a circle.

With that the probability a point lies on a sector of size $2\pi\Delta$ radian would be $\Delta \in [0,\ \ 1]$

Typically $\text{Im}(\gamma)$ an arc, if $p^{-1}\left(\text{Im}(\gamma)\right) = [\theta_1, \theta_2]$ then $\delta\left(\text{Im}(\gamma)\right) = \delta\text{o}p([\theta_1,\theta_2]) = \mu([\theta_1,\theta_2]) = \theta_2 - \theta_1$ .

Now let's consider: $p^{-1} : \mathbb{R}/\mathbb{Z} \longrightarrow [0, 1[$. $p^{-1}$ is bijective, and continuous on $p(]0, 1[)$.

$p^{-1}$ is continuous on $p(]0, 1[)$: Let $K$ be a compact in $p(]0, 1[)$. $p^{-1}(K)$ closed, bounded thus compact. for $F$ a closed set $\subset p^{-1}(K)$, $F$ compact, $\varphi^{-1}\text{o}\phi(F) = p(F)$ compact, thus closed. Hence, $p^{-1}$ is continuous on any compact of $p(]0, 1[)$. Let be $x \in p(]0, 1[)$, $p^{-1}(x) \in ]0, 1[$, there exists a compact $K$ such that $p^{-1}(x) \in K \subset ]0, 1[$, $x \in p(K)$ compact, $p^{-1}$ continuous in $x$.

Henceforth, $\delta\text{o}p$ is a measure on the measurable space $([0, 1[, \mathcal{T}_1)$, $\quad \mathcal{T}_1 = \mathcal{T}_0 \cap [0, 1[$ [2], $\mathcal{T}_0$ the Borel $\sigma$-algebra.

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1

As a matter of fact, usually we define $\varphi$ by $\varphi\text{o}\Pi \underset{\text{def.}}{=} \phi$ and prove that $\varphi$ is a homeomorphism.

2

$\mathcal{T}_0 \cap [0, 1[ \underset{\text{def.}}{=} \left\{A \cap [0, 1[, \text{ with: } A \in \mathcal{T}_0\right\}$, and we verify that $\mathcal{T}_1$ is a $\sigma$-algebra.

$\delta o p = \mu_{|[0,\ 1[}$, that is, $A$ a borel of $\mathbb{R}$: $\mu_{|[0,\ 1[}\left(A \cap [0,\ 1[\right) = \mu\left(A \cap [0,\ 1[\right) = \int_A 1_{[0,\ 1[}\, d\mu$, what we generally write for $A \in [0,\ 1[$: $\int_A d\mu$, $\int_A d\mu(x)$ or simpler $\int_A dx$ (Note that if $A$ is an interval, all this coincides with Riemmanian integral $\int_A dx$).

Thus $\delta o p$ is of unit density $1_{[0,\ 1[}$, and the probability space: $\left([0,1[,\ \mathcal{T}_1,\ \mu_{|[0,\ 1[}\right)$, describes also the odds of the experiment consisting of picking randomly a point on a circle. And is equivalent to the previous one (If we have an arc $\mathrm{Im}(\gamma)$ then $p^{-1}(\mathrm{Im}(\gamma))$ will have the same measure in the latter probability space).
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## IV Calculation example 1.

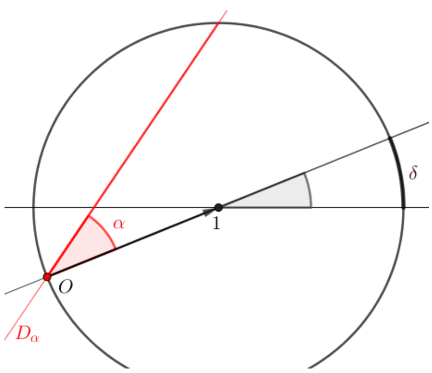
$\|.\|$ designates the Euclidean norm.
The unit circle $S_1 = \left\{u \in \mathbb{R}^2 / \|u\| = 1\right\}$ is homeomorphic to $\mathbb{R}/\mathbb{Z}$. On $\mathbb{R}/\mathbb{Z}$ we use the measure $\delta$ of density 1, which corresponds to the length of an arc divided by $2\pi$ modulo 1, on $\mathbb{R}/\mathbb{Z}$. So $\delta\left(\mathbb{R}/\mathbb{Z}\right) = \int_{\mathbb{R}/\mathbb{Z}} d\delta = 1$ . (See above chapter III on page 2 for a formal definition)

We consider the unit disk $\mathcal{D}_1^*$ of center $(1,\ 0)$ in the affine Euclidean plane with the canonical basis and for it, we define a chord as the intersection $\mathcal{D}_1^* \cap D_\alpha$ where $D_\alpha$ is a straight line whose equation in the plane is $y = \tan(\alpha)x$, if $\alpha \in \left[0,\ \frac{\pi}{2}\right[ = I_1$, and $D_{\frac{\pi}{2}} \underset{\text{def.}}{=} \left\{(x,\ \ y) \in \mathbb{R}^2, x = 0\right\}$.

So basically a chord is an element of $\Omega = \overline{I_1} \mathrm{x} \mathbb{R}/\mathbb{Z}$ (See figure: 2), let $T$ be the probability space $(\Omega,\ \ \mathcal{P}(\Omega),\ \ P)$ where we choose $P$ as the product probability of constant density hence of density $\frac{2}{\pi} 1_\Omega$, (If $x \in \Omega$ then $1_\Omega(x) = 1$ and $= 0$ otherwise) which using Fubbini since the measure $P$ is finite, yields:

$$P\left(\left[0,\ \ \frac{\pi}{6}\right] \mathrm{x} \mathbb{R}/\mathbb{Z}\right) = \int_{\left[0,\ \ \frac{\pi}{6}\right] \mathrm{x} \mathbb{R}/\mathbb{Z}} dP = \frac{2}{\pi} \int_{\left[0,\ \ \frac{\pi}{6}\right]} d\alpha \int_{\mathbb{R}/\mathbb{Z}} d\delta = \frac{1}{3}$$

Figure 2: A chord defined by an angle



# V Calculation example 2.

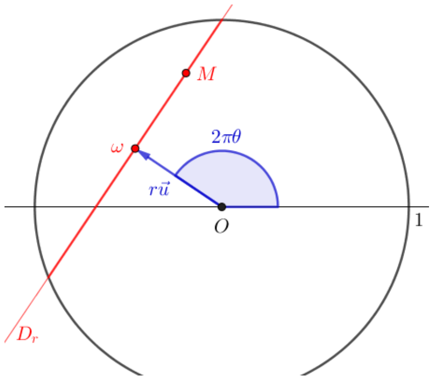
$\|.\|$ designates the Euclidean norm.
The unit circle $S_1 = \{u \in \mathbb{R}^2 / \|u\| = 1\}$ is homeomorphic to $\mathbb{R}/\mathbb{Z}$. On $\mathbb{R}/\mathbb{Z}$ we use the measure $\delta$ of density 1, which corresponds to the length of an arc divided by $2\pi$ modulo 1, on $\mathbb{R}/\mathbb{Z}$. So $\delta(\mathbb{R}/\mathbb{Z}) = \int_{\mathbb{R}/\mathbb{Z}} d\delta = 1$. (See above chapter III on page 2 for a formal definition)

We consider the unit disk $\mathscr{D}_1$ of center $O = (0,\ 0)$ in the affine Euclidean plane with the canonical basis. And for $r \in [0,\ 1] = I$ we define a chord as the intersection $\mathscr{D}_1 \cap D_r$ where $D_r$ is a straight line whose equation in the plane is $\overrightarrow{\omega M}.\vec{u} = 0$ with $\vec{u} = (\cos(2\pi\theta),\ \sin(2\pi\theta)) \in S_1 \sim \mathbb{R}/\mathbb{Z}$, $\omega = O + r\vec{u}$.

So basically a chord is an element of $\Omega = I\text{x}\mathbb{R}/\mathbb{Z}$ (See figure: 3 on the next page), let $T$ be the probability space $(\Omega,\ \mathscr{P}(\Omega),\ P)$ where we choose $P$ as the product probability of constant density hence of density $1_\Omega$, (If $x \in \Omega$ then $1_\Omega(x) = 1$ and $= 0$ otherwise) which using Fubbini since the measure $P$ is finite, yields:

$$P\left(\left[0,\ \tfrac{1}{2}\right]\text{x}\mathbb{R}/\mathbb{Z}\right) = \int_{\left[0,\frac{1}{2}\right]\text{x}\mathbb{R}/\mathbb{Z}} dP = \int_{\left[0,\frac{1}{2}\right]} dr \int_{\mathbb{R}/\mathbb{Z}} d\delta = \frac{1}{2}$$

Figure 3: a chord defined by a radius and an angle




# VI Calculation example 3.

A chord is defined as a point in a unit disk (See figure: 4 on the following page), and the event we look for is $\mathscr{D}_{\frac{1}{2}} = \{\|u\| \leq \frac{1}{2}\ /\ u \in \mathbb{R}^2\}$, $\|.\|$ the Euclidean norm, considering the probability space $(\mathscr{D}_1,\ \mathscr{P}(\mathscr{D}_1),\ P)$ where we choose $P$ as the probability of constant density hence of density $\frac{1}{\pi}1_{\mathscr{D}_1}$ (If $x \in \mathscr{D}_1$ then $1_{\mathscr{D}_1}(x) = 1$ and $= 0$ otherwise) renders:

$$P\left(\mathscr{D}_{\frac{1}{2}}\right) = \int_{\mathscr{D}_{\frac{1}{2}}} \frac{1}{\pi} dxdy = \left(\frac{1}{2}\right)^2$$

# VII Calculation example 4.

$\|.\|$ designates the Euclidean norm.
The unit circle $S_1 = \{u \in \mathbb{R}^2 / \|u\| = 1\}$ is homeomorphic to $\mathbb{R}/\mathbb{Z}$. On $\mathbb{R}/\mathbb{Z}$ we use the measure $\delta$ of density 1, which corresponds to the length of an arc divided by $2\pi$ modulo 1, on $\mathbb{R}/\mathbb{Z}$. So $\delta(\mathbb{R}/\mathbb{Z}) = \int_{\mathbb{R}/\mathbb{Z}} d\delta = 1$. (See above chapter III on page 2 for a formal definition)

We define a chord (of length smaller than 2) in a unit disk as a couple (length, orientation) in
$\Omega = [0,\ 2[\text{x}S_1 = [0,\ 2[\text{x}\mathbb{R}/\mathbb{Z}$ (See figure: 5 on the following page).

Figure 4: a chord defined by a point in a disk

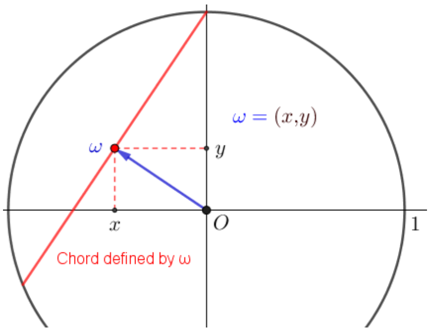



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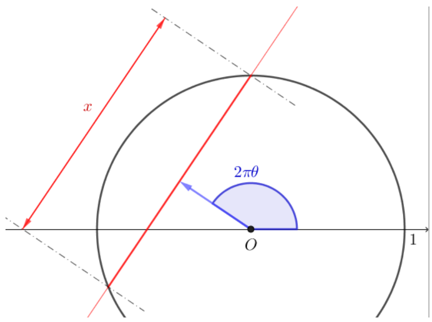
Figure 5: Drawing a chord with orientation and length




So basically a chord is an element of $\Omega = [0,\ \ 2[\text{x}\mathbb{R}/\mathbb{Z}$.
Let $T$ be the probability space $(\Omega,\ \mathscr{P}(\Omega),\ P)$ where we choose $P$ as the product probability of constant density hence of density $\frac{1}{2}1_\Omega$, (If $x \in \Omega$ then $1_\Omega(x) = 1$ and $= 0$ otherwise) which using Fubbini since the measure $P$ is finite, yields:

$$P\left(\left[\sqrt{3},2\right[\text{x}\mathbb{R}/\mathbb{Z}\right) = \int_{\left[\sqrt{3},2\right[\text{x}\mathbb{R}/\mathbb{Z}} dP = \frac{1}{2}\int_{\left[\sqrt{3},2\right[} dx \int_{\mathbb{R}/\mathbb{Z}} d\delta = 1 - \frac{\sqrt{3}}{2}$$

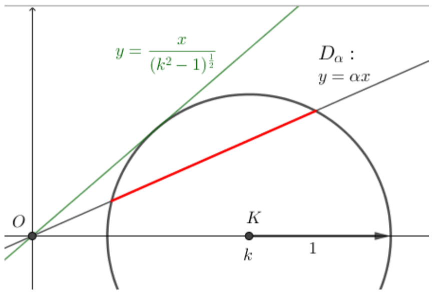
Since the measure of diameters in the disk are of null measure for the Borel-Lebesgue measure we can consider $\Omega = [0,\ \ 2]\text{x}S_1$ instead with exactly the same result.

# VIII Calculation example 5.

Let's choose a probability $q \in \left]0,\ \ \frac{1}{2}\right[$. $\|.\|$ designates the Euclidean norm.
The unit circle $S_1 = \left\{u \in \mathbb{R}^2 / \|u\| = 1\right\}$ is homeomorphic to $\mathbb{R}/\mathbb{Z}$. On $\mathbb{R}/\mathbb{Z}$ we use the measure $\delta$ of density 1, which corresponds to the length of an arc divided by $2\pi$ modulo 1, on $\mathbb{R}/\mathbb{Z}$. So $\delta\left(\mathbb{R}/\mathbb{Z}\right) = \int_{\mathbb{R}/\mathbb{Z}} d\delta = 1$. (See above chapter III on page 2 for a formal definition)

Let's consider a unit disk $\mathscr{D}_k^*$ of center $K = (k,\ \ 0),\ \ k \in ]1,\ \ +\infty[$ in the Euclidean plane with the canonical basis, and a straight line $D_\alpha$ whose equation is $y = \alpha x$ where $\alpha \in \left[0,\ \ \frac{1}{(k^2-1)^2}\right] = I_k$, then we define a chord as the intersection $\mathscr{D}_k^* \cap D_\alpha$ for a given $\alpha \in I_k$, so basically in the Bertrand's experiment a chord is an element of $\Omega_k = I_k\text{x}\mathbb{R}/\mathbb{Z}$ (See figure: 6), let $T$ be the probability space $(\Omega_k,\ \mathscr{P}(\Omega_k),\ P)$ where we choose $P$ as the product probability of constant density hence of density $\frac{1}{\mu(I_k)}1_{\Omega_k}$ (If $x \in \Omega_k$ then $1_{\Omega_k}(x) = 1$ and $= 0$ otherwise) where $\mu$ is the Borel-Lebesgue measure.

Figure 6: A chord defined as a gradiant




Let $[M_1,\ M_2]$ be a chord $\mathscr{D}_k^* \cap D_\alpha$, We consider the function $f_k$ which maps $\alpha \in I_k$ to $f_k(\alpha) = M_1M_2 \underset{\text{def.}}{=} \mathrm{d}(M_1, M_2)$ (d the Euclidean distance), $f_k$ is continuous and strictly decreasing on $I_k$,

Calculation of $f_k$ see footnote: [3]

And the probability we look for is:
$Q = P\left(f_k^{-1}([\sqrt{3},\ 2])\mathrm{x}\mathbb{R}/\mathbb{Z}\right) = \int_{\left[0,\ f_k^{-1}\left(\sqrt{3}\right)\right]\mathrm{x}\mathbb{R}/\mathbb{Z}} dP = g(k)$

Where $g$ is a homeomorphism (See calculation of $g$ in footnote: [4] which maps $]1,\ +\infty[$ to $]0,\ \frac{1}{2}[$.
So it suffices to take $k = g^{-1}(q)$ to have the probability of Bertrand's experiment be q.

# IX Calculation example 6

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Same example as example 1. but this time the straight line $D_\alpha$ is a straight line whose equation in the plane is $y = \alpha x$ where $\alpha \in \mathbb{R}_+$, instead (Notice that the chords $\{O\}\mathrm{x}\mathbb{R}/\mathbb{Z}$ are of null measure). So basically a chord is an element of $\mathbb{R}_+\mathrm{x}\mathbb{R}/\mathbb{Z}$. Now the density for $P$ can't be constant anymore because: $\int_{\mathbb{R}_+\mathrm{x}\mathbb{R}/\mathbb{Z}} d\alpha d\delta = +\infty$.

- If we choose the density equal to $\frac{2}{\pi}\frac{1}{1+\alpha^2}$ then the probability we look for is:

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[3] Calculation of $f_k$

$M_j = K + \exp(i\theta_j) = (x_j,\ \alpha x_j)$ for $j = 1$ or $2$ with $i^2 = -1$, $\mathbb{R}^2 \sim \mathbb{C}$. $x_j$ are the roots of the polynomial: $(\alpha X)^2 + (X-k)^2 - 1 = (1+\alpha^2)X^2 - 2kX + k^2 - 1$ whose half discriminant is: $\Delta = \left(k^2 + (1-k^2)(1+\alpha^2)\right)^{\frac{1}{2}} = \left(1 + (1-k^2)\alpha^2\right)^{\frac{1}{2}}$ We posit $\tan^{-1}(\alpha) = \theta$( Thus $\cos(\theta) = \frac{1}{(1+\alpha^2)^{\frac{1}{2}}}$), we have: $\frac{\sin(\theta_j)}{OM_j} = \frac{\sin(\theta)}{1}$ henceforth $M_1M_2 = |OM_1 - OM_2| = \frac{1}{|\sin(\theta)|}|y_2 - y_1| = \frac{\alpha}{\sin(\theta)}|x_2 - x_1| = (1+\alpha^2)^{\frac{1}{2}} 2\frac{\Delta}{1+\alpha^2}$, result valid even for $\theta = 0$ which leads to: $f_k(\alpha) = \frac{2\left(1+(1-k^2)\alpha^2\right)^{\frac{1}{2}}}{(1+\alpha^2)^{\frac{1}{2}}}$

The end of calculation of $f_k$

[4] Calculation of $g$

Since we have chosen equiprobability on $\Omega_k = I_k\mathrm{x}\mathbb{R}/\mathbb{Z}$, the probability we look for is $Q = \alpha(k^2-1)^{\frac{1}{2}}$ where $\alpha$ verifies $f_k(\alpha) = \sqrt{3}$ wich yields: $f_k(\alpha) = \frac{2\left(1+(1-k^2)\alpha^2\right)^{\frac{1}{2}}}{(1+\alpha^2)^{\frac{1}{2}}} = \sqrt{3}$ ,

$4\frac{1-Q^2}{1+\frac{Q^2}{k^2-1}} = 3,\ \ 1 - 4Q^2 = \frac{3Q^2}{k^2-1},\ \ Q^2 = \frac{k^2-1}{4k^2-1},\ g(k) = \left(\frac{k^2-1}{4k^2-1}\right)^{\frac{1}{2}}$

The end of calculation of $g$

$$Q = P\left(\left[0,\ \tfrac{1}{\sqrt{3}}\right]\mathrm{x}\mathbb{R}/\mathbb{Z}\right) = \int_{\left[0,\frac{1}{\sqrt{3}}\right]\mathrm{x}\mathbb{R}/\mathbb{Z}} dP = \int_{\left[0,\frac{1}{\sqrt{3}}\right]} \frac{2}{\pi}\frac{1}{1+\alpha^2}d\alpha \int_{\mathbb{R}/\mathbb{Z}} d\delta = \frac{2}{\pi}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3}$$

- If we choose the density equal to $\frac{1}{\cosh^2(\alpha)}$ then the probability we look for is:
$$Q = P\left(\left[0,\ \tfrac{1}{\sqrt{3}}\right]\mathrm{x}\mathbb{R}/\mathbb{Z}\right) = \int_{\left[0,\frac{1}{\sqrt{3}}\right]\mathrm{x}\mathbb{R}/\mathbb{Z}} dP = \int_{\left[0,\frac{1}{\sqrt{3}}\right]} \frac{1}{\cosh^2(\alpha)}d\alpha = \tanh\left(\frac{1}{\sqrt{3}}\right) \simeq 0.52$$

- If we choose the density equal to $2\exp(-\pi\alpha^2)$ then the probability we look for is:
$$Q = P\left(\left[0,\ \tfrac{1}{\sqrt{3}}\right]\mathrm{x}\mathbb{R}/\mathbb{Z}\right) = \int_{\left[0,\frac{1}{\sqrt{3}}\right]\mathrm{x}\mathbb{R}/\mathbb{Z}} dP = \int_{\left[0,\frac{1}{\sqrt{3}}\right]} 2\exp(-\pi\alpha^2)d\alpha = \frac{2}{\sqrt{2\pi}}\int_{\left[0,\frac{\sqrt{2\pi}}{\sqrt{3}}\right]} \exp\left(\frac{-x^2}{2}\right)dx$$
$$= P\left(|X| \le \sqrt{\tfrac{2\pi}{3}}\right) \simeq 0.84$$ Where $X$ is the standardized normal law.

And so on…

## X A broader view of the issue

Let's clarify.

Let's assume that we have a probability space $T = (\Omega,\ \mathcal{T},\ P)$ which describes the Bertrand's experiment, where $\Omega$, is the sample set of the chords, and $A \in \mathcal{T}$ the event: "To pick up randomly a chord of length greater than p from a unit disk", $p = \sqrt{3}$.

Since basically a chord is a length and an orientation (See figure: 5 on the previous page) we expect to define a chord with two parameters, so that we may have a measurable function, $X$: $\Omega \xrightarrow{X} \Omega_X$ with $\Omega_X$ subset of $\mathbb{R}^2$. So the random variable $X$, will have the law $P_X \underset{\text{def.}}{=} P\mathrm{o}X^{-1}$ on a $\sigma$-algebra $\mathcal{T}_X$[5]. Then somewhere we hope to find a random variable $\varphi$ mapping the length of the chord, $\varphi$: $\Omega_X \xrightarrow{\varphi} [0,\ 2]$. The probability we look for is:

$P(A) = \int_A dP = \int_{X(A)} dP_X$, with $X(A) = \varphi^{-1}([p,2])$

So in that context no matter what our choice for $X$ is, the result $P(A)$ independent of $X$ would be the same (Unless $T$ actually doesn't exist).

However if we suppose $P_X$ to have a constant density such as in examples 1. to 5. that is of density equal to $\frac{1}{\mu(\Omega_X)}1_{\Omega_X}$[6], ($dP_X = \frac{1}{\mu(\Omega_X)}1_{\Omega_X}d\mu$, $\mu$ typically the Borel-Lebesgue measure) which may in turn, allows us to calculate easily the last member of the equality above such as $\mu\big(X(A)\big)/\mu(\Omega_X)$ what we have performed in examples 1. to 5. above, then we might get in serious trouble (Not to mention that $\frac{1}{\mu(\Omega_X)}$ might be null, see example 6. above chapter IX on the preceding page) . Because we have no guarantee, nor a shred of evidence this actually be the case. Below is a closer look at what is at stake.

## XI Examples 2. or 1. vs example 3.

Let's assume we have the following commutative diagram:

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5

We may choose for $\mathcal{T}_X$: $\{E \in \mathcal{P}(X(\Omega)) / X^{-1}(E) \in \mathcal{T}\}$ which is the smallest $\sigma$-algebra rendering $X$ measurable.

6

If $x \in \Omega_X$ then $1_{\Omega_X}(x) = 1$ and $= 0$ otherwise

$$\begin{tikzcd}
 & \Omega_2 \subset \mathbb{R}^n \arrow[dd, "\psi"] \arrow[dr, "\varphi_2"] & \\
\Omega \arrow[ur, "X_2"] \arrow[dr, "X_1"'] & & K \subset \mathbb{R} \\
 & \Omega_1 \subset \mathbb{R}^n \arrow[ur, "\varphi_1"'] &
\end{tikzcd}$$

Where, for $j = 1$ or $j = 2$,

- $(\Omega,\ \mathscr{T},\ P)$ a probability space,
- $\Omega_j \subset \mathbb{R}^n, \quad K \subset \mathbb{R}, \quad n$ integer
- $(\Omega_j,\ \mathscr{T}_j,\ P_{X_j})$ a probability space,
- $\mathscr{T}_j \underset{\text{def.}}{=} \mathscr{T}_0 \cap \Omega_j,\ \ \mathscr{T}_0$ the Borel $\sigma$-algebra
- $X_j$ a random variable
- $\varphi_j$ surjective, measurable for the Borel-Lebesgue measure and the Borel $\sigma$-algebra associated with
- $\psi$ differentiable on an open set $U \subset \Omega_2$, bijective, whose Jacobian ($= \det\big(\mathrm{D}(\psi)\big)$, $\mathrm{D}(\psi)$ the differential) is of constant sign.

Assuming that $P_{X_j} \underset{\text{def.}}{=} P \mathrm{o} X_j^{-1}$ has a density $f_j$, then for any $B_2 \in \mathscr{T}_2$, $B \underset{\text{def.}}{=} X_2^{-1}(B_2) \in \mathscr{T}$:

$P(B) = \int_B dP = \int_{X_1(B)} f_1 = \int_{X_2(B)} f_2$

and:

$\int_{X_1(B)} f_1 = \int_{\psi \mathrm{o} X_2(B)} f_1 = \int_{X_2(B)} \left|\det(\mathrm{D}(\psi))\right| f_1 \mathrm{o} \psi, \qquad$ if $\Omega_2 \setminus U$ of null measure.

Where $\mathrm{D}(\psi)$ is the differential of $\psi$, hence:

$\left|\det\left(\mathrm{D}(\psi)\right)\right| f_1 \mathrm{o} \psi = f_2 \qquad \mu$-almost everywhere, if $\Omega_2 \setminus U$ of null measure.

Furthermore, $\Gamma$ a borel of $K$:

$\int_{\varphi_1^{-1}(\Gamma)} f_1 = \int_{\varphi_2^{-1}(\Gamma)} f_2$ [7], in particular,

for $K = [0,\ 2]$, $P(A) = \int_{\varphi_1^{-1}([\sqrt{3},\ 2])} f_1 = \int_{\varphi_2^{-1}([\sqrt{3},\ 2])} f_2$

## Application example 2. vs example 3.

- Example 3: $\Omega_1 = \mathscr{D}_1$, density law equal to $f_1$, $\varphi_1 : \begin{cases} \mathscr{D}_1 & \to [0,2] \\ (x,\ y) & \mapsto 2(1-x^2-y^2)^{\frac{1}{2}} \end{cases}$
- Example 2: $\Omega_2 = [0,\ 1]\text{x}[0,\ 1[$, density law equal to $f_2$, $\varphi_2 : \begin{cases} [0,\ 1]\text{x}[0,\ 1[ & \to [0,2] \\ (r,\ \theta) & \mapsto 2(1-r^2)^{\frac{1}{2}} \end{cases}$
- $K = [0,\ 2]$, $\varphi_j$ gives the length of the chord.
- $\psi : \begin{cases} [0,\ 1]\text{x}[0,\ 1[ & \to \mathscr{D}_1 \\ (r,\ \theta) & \mapsto (r\cos(2\pi\theta),\ r\sin(2\pi\theta)) \end{cases}$

$\text{D}(\psi)|_{(r,\ \theta)} = \begin{pmatrix} \cos(2\pi\theta) & -2\pi r\sin(2\pi\theta) \\ \sin(2\pi\theta) & 2\pi r\cos(2\pi\theta) \end{pmatrix}$, $\left|\det(\text{D}(\psi))\right| = 2\pi r$, $\quad U = ]0,\ 1[^2$

Thus $f_2 = 2\pi r f_1 \text{o}\psi$ $\qquad \mu$-almost everywhere.

Now if we choose $f_1$ to be constant on $\mathscr{D}_1$ that is $f_1 = \frac{1}{\pi}1_{\mathscr{D}_1}$ then
$f_2 = 2\pi r\frac{1}{\pi}1_{\psi^{-1}(\mathscr{D}_1)} = 2r1_{[0,\ 1]\text{x}[0,\ 1[}$ $\qquad \mu$-almost everywhere, which might challenge the say that

"Obviously, in example 2, the density of $P_{X_2}$ is a unit constant i.e $f_2 = 1_{[0,\ 1]\text{x}[0,\ 1[}$, thanks to equiprobability".
Especially, if we have already used the same rationale to say that the density $f_1$ for $P_{X_1}$ is constant.

P.S. : $\qquad \int_{[0,\ \frac{1}{2}]\text{x}[0,\ 1[} 2rdrd\theta = \frac{1}{4}\ \left( = \int_{\varphi_1^{-1}([\sqrt{3},\ 2])} f_1 = \int_{\mathscr{D}_{\frac{1}{2}}} \frac{1}{\pi}dxdy \right)$

## Application example 2. vs example 1.

- Example 1: $\Omega_2 = [0,\ \frac{\pi}{2}]\text{x}[0,\ 1[$, density law equal to $f_2$, $\varphi_2 : \begin{cases} [0,\ \frac{\pi}{2}]\text{x}[0,\ 1[ & \to [0,\ 2] \\ (\alpha,\ v) & \mapsto 2\cos(\alpha) \end{cases}$
- Example 2: $\Omega_1 = [0,\ 1]\text{x}[0,\ 1[$, density law equal to $f_1$, $\varphi_1 : \begin{cases} [0,\ 1]\text{x}[0,\ 1[ & \to [0,\ 2] \\ (r,\ \theta) & \mapsto 2(1-r^2)^{\frac{1}{2}} \end{cases}$

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7

Let $\Gamma$ be a borel of $K$. $B_2 \underset{\text{def.}}{=} \varphi_2^{-1}(\Gamma)$, $B \underset{\text{def.}}{=} X_2^{-1}(B_2)$

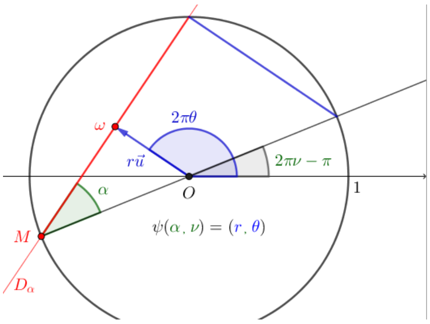
- $B_1 \underset{\text{def.}}{=} \psi(B_2)$, $X_1^{-1}(B_1) = X_2^{-1}(\psi^{-1}(B_1)) = B$
- $B_1 = \psi\text{o}\varphi_2^{-1}(\Gamma) = \varphi_1^{-1}(\Gamma)$

$P(B) = \int_{X_1(B)} f_1 = \int_{B_1} f_1$
$P(B) = \int_{X_2(B)} f_2 = \int_{B_2} f_2$

- $K=[0,\ 2]$, $\varphi_j$ gives the length of the chord.

- $\psi : \begin{cases} [0,\ \frac{\pi}{2}]\text{x}[0,\ 1[ & \to [0,\ 1]\text{x}[0,\ 1[ \\ (\alpha,\ \nu) & \mapsto \left(\sin(\alpha),\ \nu+\frac{\alpha}{2\pi}-\frac{1}{4}\right) \end{cases}$
  (See figure: 7).

Figure 7: The $\psi$ function




$\psi$ calculation details, see footnote: (8)

$\mathrm{D}(\psi)_{|(\alpha,\ \nu)}=\begin{pmatrix}\cos(\alpha) & 0\\ \frac{1}{2\pi} & 1\end{pmatrix}$, $\left|\det(\mathrm{D}(\psi))\right|=\cos(\alpha)$ $\quad U=]0,\ \frac{\pi}{2}[\text{x}]0,\ 1[$

$f_2=\cos(\alpha)f_1\text{o}\psi$ $\mu$-almost everywhere.

Now if we say that the density $f_1$ is constant thus equal to $\mathbf{1}_{[0,\ 1]\text{x}[0,\ 1[}$, such as in example 2. then it renders for example 1. of density $f_2$, without a shadow of a doubt:

$$\int_{[0,\ \frac{\pi}{6}]\text{x}[0,\ 1[} f_2=\int_{[0,\ \frac{\pi}{6}]\text{x}[0,\ 1[}\cos(\alpha)f_1\text{o}\psi=\int_{[0,\ \frac{\pi}{6}]\text{x}[0,\ 1[}\cos(\alpha)d\alpha d\nu=\frac{1}{2}$$

## XII A way out

One way out is to define a chord as a choice of two points on a circle, because assuming for each of them that they follow a unit density probability is not so far-fetched since this latter depends on the structure of the circle only, and not how we sketch up a chord whatsoever.

Let's go back to our Bertrand's conundrum.
So we posit a chord to be a choice of two points onto the circle $S_1$, so basically a chord is an element of $[0,1[^2/G$, where $G$ is the group $\{I_d,s\}$, $s$ the symmetry of $\mathbb{R}^2$ of axis $(x=y)$ in the canonical basis. We will name $\Pi_G$ the canonical projection onto $[0,1[^2/G$ (9).

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8

- For $r(\alpha,\nu)$: Thales.
- For $\theta(\alpha,\ \nu)$:$M$ on the circle and $D_\alpha$, $O$ center of the circle, $\vec{u}=\exp(2\pi i\theta)$, $\omega=O+r\vec{u}$, $\overrightarrow{\omega M}.\vec{u}=0$. $(\overrightarrow{\omega O}+\overrightarrow{OM}).\vec{u}=0$, $\overrightarrow{O\omega}.\vec{u}=r\rightsquigarrow\overrightarrow{OM}.\vec{u}=r$, $\overrightarrow{OM}=\exp(2\pi i\nu)$, $\cos\left(2\pi(\theta-\nu)\right)=r=\sin\alpha\rightsquigarrow\theta(\alpha,\ \nu)=\nu+\frac{\alpha}{2\pi}-\frac{1}{4}$.

9

$G$ acts on $[0,1[^2$ like: for $g\in G$, $x\in[0,1[^2$ $g.x\underset{\text{def.}}{=}g(x)$, the orbit of $x$ is $G.x=\{x,s(x)\}$.

Assuming that the random variables for the two points, $\theta, \nu$: $\mathbb{R}/\mathbb{Z} \longrightarrow [0,1[$ are independent, the probability space for the random variable $(\theta, \nu)$ is the product probability space $T = ([0,\ 1[^2, \mathscr{T}_1^2, \mu_{|_{[0,\ 1[^2}})$ (See chapter III on page 2). And a suitable probability space describing the Bertrand's experiment is:
$T_G = \left([0,\ 1[^2/G, G.\mathscr{T}_1^2, \mu_{|_{[0,\ 1[^2}G}\right)$[(10)]

Now we have the commutative diagram:

$$
\begin{tikzcd}
{[0,\ 1[^2} \arrow[dd, "\Pi_G"'] \arrow[dr, "\varphi"] & \\
& {[0,\ 2]} \\
{[0,\ 1[^2/G} \arrow[ur, "\varphi_G"'] &
\end{tikzcd}
$$

where:

- $$\varphi : \begin{array}{l} [0,\ 1[^2 \rightarrow [0,2] \\ (\theta,\nu) \ \mapsto \|u - v\| = 2\left|\sin\left(\pi(\theta - \nu)\right)\right| \end{array}$$

  With:
  $u = \exp(2\pi i\theta), \quad v = \exp(2\pi i\nu)$, with $i^2 = -1$, $\mathbb{R}^2 \sim \mathbb{C}$ (See figure: 8).

- $\varphi_G \text{o} \Pi_G \underset{\text{def.}}{=} \varphi$ ($\varphi_G$ is well defined since $\varphi(\theta,\ \nu) = \varphi(\nu,\ \theta), \theta,\ \nu \in [0,\ 1[)$, $\varphi_G$ is measurable.[11]

What we look for is the value of : $Q = \int_{\varphi_G^{-1}\left(\left[\sqrt{3},2\right]\right)} d\mu_G = \int_{\Pi_G \text{o} \varphi^{-1}\left(\left[\sqrt{3},2\right]\right)} d\mu_G = \int_{\varphi^{-1}\left(\left[\sqrt{3},2\right]\right)} d\mu$
So the Bertrand's event we are looking for is $\varphi^{-1}([\sqrt{3},2])$ in $T$ that is the event: $|\theta - \nu| \in B = \left[\alpha,\ 1-\alpha\right]$ with $\alpha = \frac{1}{\pi}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{3}$ [(12)].

The random variable $\theta - \nu$ has for law: $\mu_{|_{[0,\ 1[}} * \mu_{|_{]-1,\ 0]}}$, where $*$ is the convolution product. Now,
$P_{\theta-\nu} = P_{\nu-\theta}$ since $\mu_{|_{[0,\ 1[}} * \mu_{|_{]-1,\ 0]}} = \mu_{|_{]-1,\ 0]}} * \mu_{|_{[0,\ 1]}}$ ($P$ designates $\mu_{|_{[0,\ 1[^2}}$, here). The probability we look for

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10

We use the $\sigma$-algebra: $G.\mathscr{T}_1^2 \underset{\text{def.}}{=} \left\{A \in \mathscr{P}\left([0,1[^2/G\right), \text{such that } \Pi_G^{-1}(A) \in \mathscr{T}_1^2\right\}$, so $\Pi_G$ by construction is measurable for the image measure $\mu_{|_{[0,\ 1[^2}G} \underset{\text{def.}}{=} \mu_{|_{[0,\ 1[^2}} \text{o} \Pi_G^{-1}$ and then a probability.

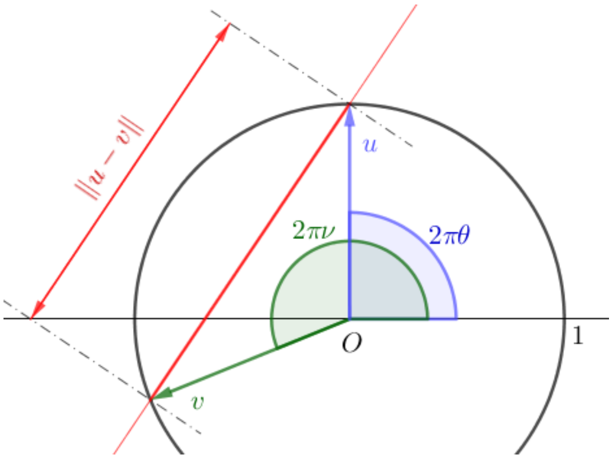
11

Let $B$ be a borel of $[0,\ 2]$, since $\varphi$ continuous, $\varphi^{-1}(B) = \Pi_G^{-1}\left(\varphi_G^{-1}(B)\right)$ is a borel, that is exactly $\varphi_G^{-1}(B) \in G.\mathscr{T}_1^2$ by definition, hence $\varphi_G$ measurable.

12

$\sqrt{3} \le \|u - v\| \le 2 \Longleftrightarrow \frac{\sqrt{3}}{2} \le \left|\sin\left(\pi(\theta - \nu)\right)\right| \le 1 \Longleftrightarrow \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \le \pi|\theta - \nu| \le \pi - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \Longleftrightarrow \alpha \le |\theta - \nu| \le 1 - \alpha$

Figure 8: A chord as two angles




is:

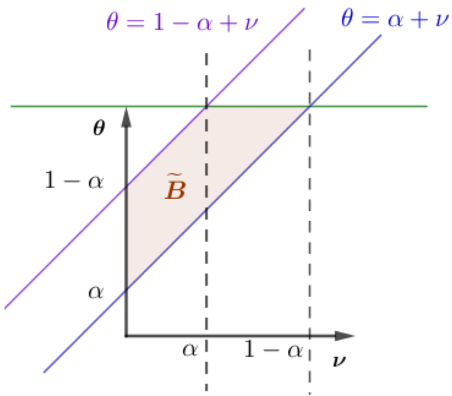
$Q = 2\int_{(\theta-\nu)\in B} d\left(\mu_{|[0,\ 1[} * \mu_{|]-1,\ 0]}\right) = 2\int_{\widetilde{B}} d\mu(\theta)\, d\mu(\nu)$, (= two times the measure of the area of $\widetilde{B}$)

where $\widetilde{B} = \left\{(\theta,\ \nu) \in [0,\ 1[^2\text{, such that } \theta - \nu \in B\right\}$ (See figure: 9 on the following page).

Using Fubini since the measures are finite leads to:

$Q = 2\int_0^{\alpha}\left(\int_{\nu+\alpha}^{1-\alpha+\nu} d\theta\right)d\nu + 2\int_{\alpha}^{1-\alpha}\left(\int_{\nu+\alpha}^{1} d\theta\right)d\nu = 2\left(\frac{1}{2}-\alpha\right) = \frac{1}{3}$ [13]

Figure 9: The $\widetilde{B}$ area




The end

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13

Area of $\widetilde{B}$ =[area of the great half square whose side length is $1-\alpha$]−[area of the small half square whose side length is $\alpha$]

$= \frac{1}{2}(1-\alpha)^2 - \frac{1}{2}\alpha^2$

$= \frac{1}{2} - \alpha$